

# Profinite Structures are Retracts of Ultraproducts of Finite Structures

*H. L. Mariano* — Centro de Lógica, Epistemologia e História da Ciência,  
Universidade de Campinas, São Paulo, S.P., Brazil.  
E-mails: hugomar@cle.unicamp.br, hugomar@ime.usp.br

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## Abstract

We establish the following model-theoretic characterization: profinite  $L$ -structures, the cofiltered limits of finite  $L$ -structures, are retracts of ultraproducts of finite  $L$ -structures. As a consequence, any elementary class of  $L$ -structures axiomatized by  $L$ -sentences of the form  $\forall \vec{x}(\psi_0(\vec{x}) \rightarrow \psi_1(\vec{x}))$ , where  $\psi_0(\vec{x}), \psi_1(\vec{x})$  are existential-positives  $L$ -formulas, is closed under the formation of profinite objects in the category **L-mod**, the category of structures suitable for the language  $L$  and  $L$ -homomorphisms.

## 1 Introduction

The results presented here belong to the interface between Category Theory and Model Theory. These results are contained in *Chapter 2* of [Mrn1]. Our primary motivation was [KMS], a paper which introduces the class of direct limits of finite abstract order spaces. The Theory of Spaces of Orderings is an axiomatization of the algebraic theory of quadratic forms on fields (see [Mar1]). Later, in [DM2], it was presented a *first-order* axiomatization of the algebraic theory of quadratic forms, the Special Groups Theory, which is, in some sense, a dual approach to the Theory of Orderings Spaces, but with an advantage: it permits an approach of quadratic forms theory by the logical methods of Model Theory.

Detailing the work:

We consider  $L$ , a first-order language with equality. We denote **L-mod**, the category of all structures suitable to the language  $L$  and  $L$ -homomorphisms. As preparation we present some species of limits and colimits in the category **L-mod** and we relate one of the principals constructions in Model Theory, the notion of reduced product of structures, with the categorial constructions of product and filtered colimit in **L-mod** (Proposition 14). Our main result, the Theorem 18, claims that the *profinite*  $L$ -structures, the cofiltered limits of finite  $L$ -structures, are *retracts of ultraproducts* of finite  $L$ -structures. As a consequence, each elementary class of  $L$ -structures axiomatized by  $L$ -sentences like  $\forall \vec{x}(\psi_0(\vec{x}) \rightarrow \psi_1(\vec{x}))$ , where  $\psi_0(\vec{x}), \psi_1(\vec{x})$  are existential-positives  $L$ -formulas, is closed under the formation of profinite objects in the category **L-mod** (Corollary 22).

Applying the central results, 18 and 22, to the Special Groups Theory we conclude that there are profinite special groups and that they are retracts of ultraproducts of finite special groups (Section 5).

## 2 Preliminaries

We assume familiarity with the basic notions of Category Theory (category, functor, natural transformation, limits/colimits, ...) and of Model Theory (language, structure, homomorphism, elementary embedding, reduced products, ...). Our reference about Category Theory is [Mac]; for Model Theory we use [CK] e [BS].

We clarify below some topics needed to the development of the results obtained in this work.

## 2.1 Retracts

Let  $\mathcal{C}$  be a category and  $A, B$  objects of  $\mathcal{C}$ .  $A$  is called a *retract* of  $B$  when there are morphisms  $s : A \rightarrow B$  and  $r : B \rightarrow A$  such that  $r \circ s = Id_A : A \rightarrow A$ . In this case we say that  $r$  is a retraction and  $s$  a section.

Is immediate to verify that any section is a monomorphism and, dually, any retraction is an epimorphism; that a morphism is invertible (or isomorphism) precisely when it is simultaneously a section and a retraction.

We remark that the proposition: “all epimorphism in the category of sets and functions (**Set**) is a retraction” is equivalent to the Axiom of Choice.

## 2.2 Directed Sets

Let  $\langle I, \leq \rangle$  be a poset, i.e.  $\leq \subseteq I \times I$  is a binary relation that is reflexive, symmetric and transitive in the set  $I$ . For each  $i \in I$  we define  $i^\leftarrow = \{j \in I : j \leq i\}$ ,  $i^\rightarrow = \{j \in I : i \leq j\}$ . We say that:

- \*  $\langle I, \leq \rangle$  is *upward directed* (or *filtered*) if  $I \neq \emptyset$  and for each  $i, j \in I$ ,  $i^\rightarrow \cap j^\rightarrow \neq \emptyset$ .
- \*  $\langle I, \leq \rangle$  is *downward directed* (or *cofiltered*) if  $I \neq \emptyset$  and for each  $i, j \in I$ ,  $i^\leftarrow \cap j^\leftarrow \neq \emptyset$ .

Clearly a poset  $\langle I, \leq \rangle$  is upward directed iff its opposite poset  $\langle I, \leq \rangle^{op}$  is downward directed and vice-versa. When we make mention to directed posets we *always* will be referring the upward directed orders.

We say that a filter  $\mathcal{F}$  in the set  $I$  is a *directed filter* in the poset  $\langle I, \leq \rangle$  when, for each  $i \in I$ , we have  $i^\rightarrow \in \mathcal{F}$ .

**Lemma 1** *If  $\langle I, \leq \rangle$  is a directed poset then there is a directed ultrafilter  $\mathcal{U}$  in  $\langle I, \leq \rangle$ .*

**Proof.** Because  $\langle I, \leq \rangle$  is directed we verify, by induction on  $n \in \omega$ , that for each  $\{i_0, \dots, i_{n-1}\} \subseteq I$  there is  $j \in I$  such that  $j \geq i_0, \dots, i_{n-1}$ , so  $\emptyset \neq j^\rightarrow \subseteq \bigcap_{m < n} i_m^\rightarrow$ . Hence the set  $S = \{i^\rightarrow : i \in I\}$  has the finite intersection property and then there is an ultrafilter  $\mathcal{U}$  such that  $S \subseteq \mathcal{U}$ .  $\square$

## 2.3 Pure Morphisms

Let  $L$  an arbitrary first-order language with equality.

**Definition 2** *A formula  $\varphi$  in the language  $L$  is called:*

- \* *positive if the symbols of implication and negation do not occur in  $\varphi$ ;*
- \* *existential positive (e.p.) if it is obtained from the atomic formulas by the connectives  $\wedge, \vee$  and the existential quantifier  $\exists$ ;*
- \* *positive primitive (p.p.) if it is written like  $\exists \bar{x} \varphi$ , where  $\varphi$  is a conjunction of atomic formulas.*

We denote:

- \*  $\exists^+(L)$  the set of all  $L$ -formulas that are logically equivalents, in the classical predicate calculus, to a formula existential positive in  $L$ ;
- \*  $\text{pp}(L)$  the set of all  $L$ -formulas that are logically equivalents, in the classical predicate calculus, to a formula positive primitive in  $L$ .

By induction on complexity, we see that if  $\varphi \in \exists^+(L)$  then there are finite subsets  $P_1, \dots, P_n$  of  $\text{pp}(L)$  such that

$$\vdash_{\mathcal{C}} \varphi \leftrightarrow (\psi_1 \vee \psi_2 \vee \dots \vee \psi_n),$$

where  $\psi_j$  is a conjunction of formulas in  $P_j$ ,  $1 \leq j \leq n$ .

**Definition 3** *A function between  $L$ -structures,  $f : M \rightarrow N$ , is called a pure  $L$ -morphism if for each formula  $\varphi(v_1, \dots, v_n) \in \exists^+(L)$  and  $\bar{a} \in M^n$*

$$M \models \varphi[\bar{a}] \leftrightarrow N \models \varphi[f\bar{a}].$$

It is not difficult to see that: a function between  $L$ -structures is a pure  $L$ -morphism iff it is a  $L$ -homomorphism that reflects the validity of formulas in  $pp(L)$ ; all pure  $L$ -morphism is a  $L$ -imbedding; all elementary  $L$ -imbedding and all  $L$ -section<sup>1</sup> is a pure  $L$ -imbedding. We register also the following

**Lemma 4** *Let  $\Sigma$  be a set of  $L$ -sentences of the form  $\forall \vec{x}(\psi_0(\vec{x}) \rightarrow \psi_1(\vec{x}))$ , where  $\psi_0(\vec{x}), \psi_1(\vec{x}) \in \exists^+(L)$ . Let  $N$  be a  $L$ -structure such that  $N \models \Sigma$ ; if  $M$  is a  $L$ -structure and there is a pure  $L$ -morphism from  $M$  to  $N$  then  $M \models \Sigma$ .  $\square$*

### 3 The category $\mathbf{L-mod}$

Henceforth we fix  $L$  an arbitrary first-order language with equality. We shall write  $ct(L)$  for the set of all symbols for constants of the language and, for each  $n \geq 1$ ,  $op(n, L)$  denotes the set of all symbols for operations with aridity  $n$  and  $rel(n, L)$  for the set of all symbols for  $n$ -ary relations.

We denote  $\mathbf{L-mod}$  the category of all structures suitable to the language  $L$  and of  $L$ -homomorphisms between them<sup>2</sup>.

$\mathbf{L-mod}$  is a complete and cocomplete category, i. e., all diagram  $D : \mathcal{I} \longrightarrow \mathbf{L-mod}$ , where  $\mathcal{I}$  is a small category, is base of some limit cone and some colimit co-cone ([Mac], Chapter 5).

We will detail below some of that categorial constructions and how the reduced products, one of the fundamentals notions of Model Theory, is related with these constructions.

#### 3.1 Limits in $\mathbf{L-mod}$

**5 Products in  $\mathbf{L-mod}$ :** Let  $I$  a set and  $\{M_i : i \in I\}$ , a family of  $L$ -structures. We consider  $M = \prod_{i \in I} M_i$  the product of their underlying sets. We make  $M$  a  $L$ -structure, defining the  $L$ -symbols interpretations coordinate-wise. More explicitly, for each natural  $n \geq 1$  :

\* If  $c \in ct(L)$ ,  $c^M = \langle c^{M_i} \rangle_{i \in I}$ ;

\* Se  $\omega \in op(n, L)$  e  $\langle s_1, \dots, s_n \rangle \in M^n$ , then

$$\omega^M(s_1, \dots, s_n) = \langle \omega^{M_i}(s_1(i), s_2(i), \dots, s_n(i)) \rangle_{i \in I};$$

\* If  $R \in rel(n, L)$  e  $\vec{s} \in M^n$ , then

$$M \models R[\vec{s}] \Leftrightarrow \forall i \in I, M_i \models R[s_1(i), s_2(i), \dots, s_n(i)].$$

By induction on the complexity of terms and formulas we get:

(A) If  $\tau(v_1, \dots, v_n)$  is a term em  $L$  e  $\vec{s} \in M^n$ , then

$$\tau^M(\vec{s}) = \langle \tau^{M_i}(s_1(i), \dots, s_n(i)) \rangle_{i \in I}.$$

(B) If  $\varphi(v_1, \dots, v_n)$  is a atomic formula in  $L$  and  $\vec{s} \in M^n$ , then

$$M \models \varphi[\vec{s}] \Leftrightarrow \forall i \in I, M_i \models \varphi[s_1(i), \dots, s_n(i)].$$

Observe that the canonicals projections,  $\pi_i : M \longrightarrow M_i$ ,  $i \in I$ , are  $L$ -morphisms. It is easy to see that this construction is the product of the family  $\{M_i : i \in I\}$  in the category  $\mathbf{L-mod}$ .

Particularly, when  $I = \emptyset$ , we have the

**Final object of  $\mathbf{L-mod}$  :** Let  $\mathbf{1} = \{\emptyset\}$ , where all  $n$ -ary relation symbols are interpreted by  $\mathbf{1}^n$ , all  $n$ -ary functional symbols are interpreted as the unique function  $\mathbf{1}^n \rightarrow \mathbf{1}$  and all constant symbols are interpreted as the unique element of  $\mathbf{1}$ . We see, by induction on the complexity, that all  $L$ -formula positive (Definition 2) is satisfiable em  $\mathbf{1}$ ; hence all  $L$ -sentence of the form  $\forall \vec{x}(\psi_0(\vec{x}) \rightarrow \psi_1(\vec{x}))$ , where  $\psi_0(\vec{x}), \psi_1(\vec{x})$  are positive  $L$ -formulas, is true in  $\mathbf{1}$ .

<sup>1</sup>A  $L$ -section is a  $L$ -homomorphism that admits a retraction that is also a  $L$ -homomorphism.

<sup>2</sup>We will *not exclude* here the possibility of a  $L$ -structure be empty. As a structure is non-empty iff it satisfies the sentence  $\exists v_0(v_0 = v_0)$  we should write the instantiation axiom as  $\forall v \varphi \wedge \exists v_0(v_0 = v_0) \rightarrow \varphi(\ulcorner \tau \urcorner \mid v \urcorner)$  were  $\tau$  is a term free for  $v$  in  $\varphi$ .

We observe also that  $\prod_{i \in I} M_i = \emptyset$  iff there is a  $i \in I$  such that  $M_i = \emptyset$ .<sup>3</sup>

**6 Equalizers in  $\mathbf{L-mod}$ :** Let  $D = (A \xrightleftharpoons[g]{f} B)$  be  $L$ -morphisms. We define

$$E = \{a \in A : f(a) = g(a)\}.$$

If  $c \in ct(L)$ ,  $c^E =_{def} c^A \in E$ . Further, if  $\omega \in op(n, L)$  and  $\bar{a} \in E^n$ , then

$$f(\omega^A(\bar{a})) = \omega^B(f(\bar{a})) = \omega^B(g(\bar{a})) = g(\omega^A(\bar{a})),$$

and  $E$  is closed with respect to functional symbols interpretations. For each  $R \in rel(n, L)$ , let  $R^E = R \cap E^n$ . So the canonical inclusion,  $\eta : E \rightarrow A$ , is a  $L$ -imbedding. Moreover, because  $f \circ \eta = g \circ \eta$ ,  $(E; \{\eta, f \circ \eta\})$  is a cone over  $D$  in  $\mathbf{L-mod}$ . This cone is the equalizer of  $(f, g)$ .

From the remarks 5 and 6 below and the construction of limits from products and equalizers ([Mac], section 5.2) we get the

**Corollary 7** Let  $D : \mathcal{I} \rightarrow \mathbf{L-mod} : (i \xrightarrow{\alpha} j) \mapsto (D_i \xrightarrow{f_\alpha} D_j)$  be a  $\mathcal{I}$ -diagram in  $\mathbf{L-mod}$  and  $(M \xrightarrow{\lambda_i} D_i : i \in Obj(\mathcal{I}))$  be a cone over  $D$ . We take  $\lambda = (\lambda_i)_{i \in I}$  the unique function from  $M$  to  $\prod_{i \in I} D_i$ , such that  $\pi_i \circ \lambda = \lambda_i$ , for each  $i \in Obj(\mathcal{I})$ . Then  $M$  is (isomorphic to)  $\varprojlim D$  iff

[lim 1] : The image of  $\lambda$  in  $\prod_{i \in I} D_i$  is the set

$$\{x \in \prod_{i \in I} D_i : \text{for all arrow of } \mathcal{I}, (i \xrightarrow{\alpha} j), \text{ we have } f_\alpha(\pi_i(x)) = \pi_j(x)\}.$$

[lim 2] : If  $\varphi(v_1, \dots, v_n)$  is an atomic formula in  $L$  and  $\bar{s} \in M^n$ ,

$$M \models \varphi[\bar{s}] \Leftrightarrow \forall i \in Obj(\mathcal{I}), D_i \models \varphi[\lambda_i(\bar{s})].$$

□

### 3.2 Reduced Products and Ultraproducts of $\mathbf{L}$ -structures

**8** Let  $I$  be a non empty set and  $\{M_i : i \in I\}$  be a family of  $L$ -structures, all non empty. We fix  $\mathcal{F}$  a filter in  $I$  and we consider  $M = \prod_{i \in I} M_i$  the product of their underlying sets (so  $M \neq \emptyset$ ). We define a binary relation  $\theta_F$  in  $M$  :

$$x \theta_F y \Leftrightarrow \{i \in I : x(i) = y(i)\} \in \mathcal{F}.$$

It is easy to check that  $\theta_F$  is an equivalence relation in  $M$ . We will write

$$M/\mathcal{F} = \{x/\mathcal{F} : x \in M\}$$

the set of all equivalence classes of  $\theta_F$  ( $M/\mathcal{F} \neq \emptyset$ ). If  $\bar{x} \in M^n$ , we define

$$\bar{x}/\mathcal{F} = \langle x_1/\mathcal{F}, \dots, x_n/\mathcal{F} \rangle \in (M/\mathcal{F})^n.$$

For each  $\bar{x} \in M^n$  and  $i \in I$ , we take

$$\bar{x}(i) = \langle x_1(i), \dots, x_n(i) \rangle \in M_i^n.$$

With the notation in 5, if  $R \in rel(n, L)$ ,  $\omega \in op(n, L)$  and  $\bar{x}, \bar{y} \in M^n$  are such that  $\bar{x}/\mathcal{F} = \bar{y}/\mathcal{F}$ , then:

(A)  $\omega^M(\bar{x})/\mathcal{F} = \omega^M(\bar{y})/\mathcal{F}$ ;

(B)  $\{i \in I : M_i \models R[\bar{x}(i)]\} \in \mathcal{F} \Leftrightarrow \{i \in I : M_i \models R[\bar{y}(i)]\} \in \mathcal{F}$ .

With the aid of (A) e (B) we can make  $M/\mathcal{F}$  a  $L$ -structure through the followings conditions:

\* If  $c \in ct(L)$ ,  $c^{M/\mathcal{F}} = \langle c^{M_i} \rangle/\mathcal{F}$ , i. e., the interpretation of the constant symbol  $c$  in  $M/\mathcal{F}$  is the equivalence class of the  $I$ -sequence whose coordinates are the interpretations of  $c$  in each component  $M_i$ ;

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<sup>3</sup>A version of choice axiom.

\* If  $\omega \in op(n, L)$  and  $\bar{x} \in M^n$ , then  $\omega^{M/F}(\bar{x}/\mathcal{F}) = \omega^M(\bar{x})/\mathcal{F}$ ;

\* If  $R \in rel(n, L)$  and  $\bar{x} \in M^n$ ,

$$M/\mathcal{F} \models R[\bar{x}/\mathcal{F}] \Leftrightarrow \{i \in I : M_i \models R[\bar{x}(i)]\} \in \mathcal{F}.$$

Induction on complexity gives

(C) If  $\tau(v_1, \dots, v_n)$  is a term in  $L$  and  $\bar{x} \in M^n$ ,  $\tau^{M/F}(\bar{x}/\mathcal{F}) = \tau^M(\bar{x})/\mathcal{F}$ .

(D) If  $\varphi(v_1, \dots, v_n)$  is an atomic formula in  $L$  and  $\bar{x} \in M^n$ ,

$$M/\mathcal{F} \models \varphi[\bar{x}/\mathcal{F}] \Leftrightarrow \{i \in I : M_i \models \varphi[\bar{x}(i)]\} \in \mathcal{F}.$$

(E) The natural map  $x \in M \mapsto x/\mathcal{F} \in M/\mathcal{F}$  is a surjective  $L$ -homomorphism.

The  $L$ -structure  $M/\mathcal{F}$  is named *the reduced product* of the family  $\{M_i : i \in I\}$  by the filter  $\mathcal{F}$ . If  $\mathcal{F}$  is an ultrafilter in  $I$ ,  $M/\mathcal{F}$  is called *the ultraproduct* of  $\{M_i : i \in I\}$  by the ultrafilter  $\mathcal{F}$ . When all the  $L$ -structures are the same,  $M_i = N$ ,  $i \in I$ , the correspondent construction is called *reduced power* and *ultrapower*, when  $\mathcal{F}$  is an ultrafilter, it is indicated  $N^I/\mathcal{F}$ .

The fundamental result concerning ultraproducts is the:

**Theorem 9** (Łós's Theorem) *Let  $I$  a non empty set,  $\{M_i : i \in I\}$  is a family of non empty  $L$ -structures,  $M = \prod_{i \in I} M_i$  and  $\mathcal{F}$  an ultrafilter in  $I$ . Then for all formula  $\varphi(v_1, \dots, v_n)$  in  $L$  and all  $\bar{x} \in M^n$*

$$(L) \quad M/\mathcal{F} \models \varphi[\bar{x}/\mathcal{F}] \Leftrightarrow \{i \in I : M_i \models \varphi[\bar{x}(i)]\} \in \mathcal{F}.$$

**Proof.** See Theorem 4.1.9, page 217, in [CK] or Theorem 5.2.1, page 90, in [BS]. □

**Remark 10** We add that the equivalence in the Łós's Theorem remains true for reduced products in general ( $\mathcal{F}$  is a filter) if we restrict ourselves to formulas  $\varphi(v_1, \dots, v_n)$  that are in  $p.p.(L)$  or, most generically, to the formulas that are generated from the atomic formulas by the usage of the conjunction and both quantifiers.

If  $M$  is a  $L$ -structure,  $I$  is a set e  $\mathcal{F} \subseteq P(I)$  is a filter in  $I$ , then there is a canonical  $L$ -homomorphism, *the diagonal from  $M$  to  $M^I/\mathcal{F}$*

$$\delta : M \longrightarrow M^I/\mathcal{F}, \text{ where } \delta(a) = \langle a \rangle/\mathcal{F},$$

for each  $a \in M$  in the equivalence class of the constant  $I$ -sequence of value  $a$ .

It follows from Łós's Theorem that when  $\mathcal{F}$  is a ultrafilter in  $I$  then the diagonal morphism,  $\delta : M \longrightarrow M^I/\mathcal{F}$ , is a elementary embedding. Similarly, if  $\mathcal{F}$  is just a (proper) filter in  $I$  then the diagonal morphism,  $\delta : M \longrightarrow M^I/\mathcal{F}$ , is a just a pure embedding (item 2.3).

Another important consequence of this Theorem is that any elementary class of structures is closed under the ultraproduct construction.

### 3.3 Colimits in $L\text{-mod}$

**11 Filtered Colimits in  $L\text{-mod}$ :** Let  $\langle I, \leq \rangle$  be a directed poset and  $\mathcal{M}$  an  $I$ -diagram <sup>4</sup> in  $L\text{-mod}$ .

$$\mathcal{M} : I \longrightarrow L\text{-mod} : (i \leq j) \mapsto (M_i \xrightarrow{f_{ij}} M_j)$$

Let  $W = \prod_{i \in I} M_i = \bigcup_{i \in I} M_i \times \{i\}$ , be the disjunct reunion of the sets  $M_i$ . We have the canonical functions  $w_i : M_i \longrightarrow W$ ,  $x \mapsto \langle x, i \rangle$ . As  $I$  is a directed poset the prescription

$$\langle x, i \rangle \equiv \langle y, j \rangle \Leftrightarrow \exists k \geq i, j \text{ such that } f_{ik}(x) = f_{jk}(y),$$

defines an equivalence relation  $\equiv$  in  $W$ . Let

$$M = \{\langle x, i \rangle / \equiv : \langle x, i \rangle \in W\}$$

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<sup>4</sup>As usual, we consider here  $I$  as a category whose objects are the members of the set  $I$  and whose arrows are the elements of the binary relation  $\leq$ .

be the set of all equivalence classes of  $\equiv$ . Notice that for each constant symbol  $c$  in  $L$  we have  $\langle c^{M_i}, i \rangle \equiv \langle c^{M_j}, j \rangle$ . We interpret  $L$  in  $M$  as follows: for each  $n \geq 1$  and  $\bar{x} \in M^n$ ,  $\bar{x} = \langle \langle x_1, i_1 \rangle / \equiv, \dots, \langle x_n, i_n \rangle / \equiv \rangle$ , we define:

(A) If  $R \in \text{rel}(n, L)$  then  $M \models R[\bar{x}]$  iff

$$\exists k \geq i_1, \dots, i_n, \text{ such that } M_k \models R[f_{i_1 k}(x_1), \dots, f_{i_n k}(x_n)].$$

(B) If  $\omega \in \text{op}(n, L)$  we take  $k \geq i_1, \dots, i_n$  and define  $\omega^M(\bar{x})$  as the equivalence class of the pair

$$\langle \omega^{M_k}(f_{i_1 k}(x_1), \dots, f_{i_n k}(x_n)), k \rangle.$$

(C) If  $c \in \text{ct}(L)$  we take  $c^M = \langle c^{M_i}, i \rangle / \equiv$ .

Because  $I$  is directed, the constructions above are independent of the particular choice of representations and also of the index choice made above. Further, the compositions of the quotient function,  $q : W \rightarrow M$ , with the functions  $w_i$ , defines  $L$ -homomorphisms  $\alpha_i : M_i \rightarrow M$  that make  $(M, \{\alpha_i : i \in I\})$  a co-cone over the diagram  $\mathcal{M}$ . This co-cone is the colimit  $\varinjlim \mathcal{M}$ .

**Corollary 12** *Let  $\mathcal{M} = (M_i, \{f_{ij} : i \leq j\})$  be an  $I$ -diagram in  $\mathbf{L-mod}$ , where  $I$  is a directed poset. A co-cone in  $\mathbf{L-mod}$  over  $\mathcal{M}$ ,  $(N, \{\beta_i : i \in I\})$ , is (isomorphic to)  $\varinjlim \mathcal{M}$  iff it verifies the following conditions:*

[colim 1] :  $N = \bigcup \{\beta_i(M_i) : i \in I\}$ .

[colim 2] : If  $\varphi(v_1, \dots, v_n)$  is an atomic formula in  $L$  and  $\bar{s} \in N^n$ ,

$$N \models \varphi[\bar{s}] \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \exists k \in I \text{ and } \bar{x} \in M_k^n \text{ such that} \\ s_p = \beta_k(x_p), 1 \leq p \leq n, \\ \text{and } M_k \models \varphi[\bar{x}]. \end{array} \right.$$

□

### 3.4 Reduced products and filtered colimits of products

There is a connection between reduced products and certain filtered colimits<sup>5</sup> that will be very useful in the proof of our main result, namely Theorem 18. Before the precise statement and its proof we need to establish some notation.

**13** Let  $L$  be a first-order language with equality,  $I$  a non-empty set,  $\{M_i : i \in I\}$  a family of  $L$ -structures all non-empty and  $M = \prod_{i \in I} M_i$  their product (item 5).

(A) For each  $J \subseteq I$  let  $M_{|J} = \prod_{j \in J} M_j$ ;

(B) If  $J \subseteq K \subseteq I$  then there is a canonical  $L$ -morphism,  $\pi_{KJ} : M_{|K} \rightarrow M_{|J}$ , that forgets the coordinates out of  $J$ , that is, for  $x \in M_{|K}$ ,  $\pi_{KJ}(x) = x_{|J}$  (we recall that  $x$  is a function from  $K$  to  $\bigcup_{k \in K} M_k$ ). Remark that

$$(*) \quad \pi_{JJ} = \text{Id}_{M_{|J}} \quad \text{and} \quad J \subseteq K \subseteq W \Rightarrow \pi_{WJ} = \pi_{KJ} \circ \pi_{WK}.$$

The canonical projections,  $\pi_i : M \rightarrow M_i$ , correspond to  $\pi_{I\{i\}}$ .

(C) For each  $J \subseteq I$  we define  $*$  :  $M_{|J} \times M \rightarrow M$ ,  $\langle s, x \rangle \mapsto s * x$ , where

$$s * x(i) = \begin{cases} s(i) & \text{if } i \in J \\ x(i) & \text{if } i \notin J. \end{cases}$$

Note that when  $J = I$  then the operation  $*$  is the projection in the first coordinate. Equivalently, for each  $x \in M$ , the function  $(\cdot) * x : M \rightarrow M$  is the identity function.

Let  $\mathcal{F}$  be a filter in  $I$ . Then  $\langle \mathcal{F}, \subseteq \rangle$  is a downward directed poset (item 2.2) because for each  $J, K \in \mathcal{F}$  then  $J \cap K \in \mathcal{F}$ . Consequently,  $\mathcal{F}^{op}$ , the opposite poset of  $\langle \mathcal{F}, \subseteq \rangle$ , is (upward) directed. Consider

$$\mathcal{M} = (M_{|J}, \{\pi_{KJ} : J \subseteq K \text{ and } J \in \mathcal{F}\}).$$

<sup>5</sup>The geometrical girth of this result appears in [Eil] and [Mir].

By (\*) in (B),  $\mathcal{M}$  is a  $\mathcal{F}^{op}$ -diagram in **L-mod**, the *directed diagram* associated to the family  $\{M_i : i \in I\}$  and to the filter  $\mathcal{F}$  in  $I$ .

**Proposition 14** *Let  $L$  be a first-order language with equality,  $I$  a non-empty set,  $\{M_i : i \in I\}$  a family of  $L$ -structures all non-empty and  $\mathcal{F}$  a filter in  $I$ . Consider*

$$\mathcal{M} = (M_{|J}, \{\pi_{KJ} : J \subseteq K, J \in \mathcal{F}\})$$

*the directed diagram associated to the family  $\{M_i : i \in I\}$  and to the filter  $\mathcal{F}$  in  $I$  as in 13. Then  $\varinjlim \mathcal{M}$  is naturally  $L$ -isomorphic to the reduced product  $\prod_{i \in I} M_i / \mathcal{F}$ .*

**Proof.** (Sketch) We make use of the notation in 13. In particular,  $M_{|I} = \prod_{i \in I} M_i$  is the  $L$ -structure product.

(A) We fix a  $t \in M$ . For each  $J \in \mathcal{F}$ , we consider the mapping  $\nu_J : M_{|J} \longrightarrow M/\mathcal{F}$ , given by

$$\nu_J(s) = (s * t) / \mathcal{F}.$$

As  $M \neq \emptyset$  this definition make sense. Follows directly from the definition of reduced product that the function  $\nu_J$  *does not depends* of the particular element  $t \in M$  chosen.

(B) It may be verified, from the constructions of the product structure, reduced product and filtered colimit that, for each  $J \in \mathcal{F}$  the function  $\nu_J : M_{|J} \longrightarrow M/\mathcal{F}$  is a  $L$ -homomorphism. Further, for each  $J, K \in \mathcal{F}$  such that  $J \subseteq K$ , the following diagram commutes:

$$\begin{array}{ccc} M_{|K} & \xrightarrow{\pi_{KJ}} & M_{|J} \\ \nu_K \searrow & & \swarrow \nu_J \\ & M/\mathcal{F} & \end{array}$$

(C) It may be checked that  $(M/\mathcal{F}, \{\nu_J : J \in \mathcal{F}\})$  is a co-cone over the diagram  $\mathcal{M}$  that satisfies a universal property and then it must be (isomorphic to) the co-cone  $\varinjlim \mathcal{M}$ , the colimit of the diagram  $\mathcal{M}$ .  $\square$

**Remark 15** We note that if  $\mathcal{M} = (M_{|J}, \{\pi_{KJ} : J \subseteq K, J \in \mathcal{F}\})$  is the directed diagram of the proposition below then the  $L$ -structure  $\varinjlim \mathcal{M}$  seems to be the “fundamental” notion of reduced product (or ultraproduct, when  $\mathcal{F}$  is a ultrafilter) because this is the structure that *always* is defined and that *always* satisfies Łós’s equivalence (Ł), page 5, and its version for reduced products (see Theorem 9 and Remark 10). However, if we admit just an empty structure  $M_j$  in the original definition of reduced product, take some filter  $\mathcal{F}$  such that  $\{j\} \notin \mathcal{F}$  and regard the p.p.-sentence “I am not the empty structure” :  $\exists v_0 (v_0 = v_0)$ , then we have that  $\prod_{i \in I} M_i / \mathcal{F}$  is empty but  $\{i \in I : M_i \text{ is empty}\} \notin \mathcal{F}$ .

## 4 Profinite Structures and Ultraproducts

We present now ours results.

**Definition 16** *A  $L$ -structure is **profinite** when it is  $L$ -isomorphic to the limit of a diagram of finite  $L$ -structures over a downward directed poset.*

**Remark 17** If  $P$  is a profinite  $L$ -structure then there is an *upward* directed poset,  $\langle I, \leq \rangle$ , and a cofiltered diagram of finite  $L$ -structures over  $I$ ,

$$\mathcal{M} = (M_i, \{f_{ji} : i \leq j\})$$

such that  $(P, \{\lambda_i : i \in I\}) = \varprojlim \mathcal{M}$ . By Proposition 7 we can consider  $P$  as a substructure of the product  $M = \prod_{i \in I} M_i$ , i.e., there is a natural  $L$ -imbedding,  $\iota : P \longrightarrow M$ , such that for all  $i \in I$ ,

$$(\sharp) \quad \lambda_i = \pi_i \circ \iota.$$

$$\begin{array}{ccc} P & \xrightarrow{\iota} & M \\ & \searrow \lambda_i & \swarrow \pi_i \\ & & M_i \end{array}$$

where  $\pi_i : M \longrightarrow M_i$  is the a canonical projection. Further, it follows from [lim 1] in 7 that

$$(b) \quad \forall \bar{x} \in P \quad \forall j, k \in I \quad (j \in k^\rightarrow \Rightarrow f_{jk}(x_j) = x_k).$$

We saw in 13 that, if  $\mathcal{F}$  is a filter in  $I$  then for each  $J \in \mathcal{F}$  we have a natural  $L$ -morphism

$$\nu_J : M_{|J} \longrightarrow M/\mathcal{F}, \text{ given by } x \longmapsto x/\mathcal{F},$$

where  $M/\mathcal{F}$  indicates the reduced product  $\prod_{i \in I} M_i/\mathcal{F}$ .

With these preliminary we enunciate the

**Theorem 18** *Profinite  $L$ -structures are retracts of ultraproducts of finite  $L$ -structures. More precisely, and with the notation in 17, let  $\langle I, \leq \rangle$  be a directed poset and*

$$\mathcal{M} = (M_i, \{f_{ji} : i \leq j\})$$

*a cofiltered diagram of finite  $L$ -structures over  $I$ . If  $\varprojlim \mathcal{M} = (P, \{\lambda_i : i \in I\})$  then the  $L$ -morphism that is the composition*

$$P \xrightarrow{\iota} \prod_{i \in I} M_i \xrightarrow{\nu_I} \prod_{i \in I} M_i/\mathcal{U},$$

*is an  $L$ -section (item 2.1), where  $\mathcal{U}$  is a directed ultrafilter in  $I$  (item 2.2) .*

$$\begin{array}{ccc} P & \xrightarrow{\iota} & \prod_{i \in I} M_i & \xrightarrow{\nu_I} & \prod_{i \in I} M_i/\mathcal{U} \\ & \searrow Id_P & & \downarrow \gamma^U & \\ & & P & & \end{array} \quad \begin{array}{c} \circlearrowleft \\ \text{Profinite and ultraproduct of finite} \end{array}$$

**Proof.** By Lemma 1 there is a directed ultrafilter in  $\langle I, \leq \rangle$ ; the proof will be carried on fixing a such ultrafilter  $\mathcal{U}$ .

Let  $M = \prod_{i \in I} M_i$  be the product  $L$ -structure of the family  $\{M_i : i \in I\}$ . By the Proposition 14 (and with the same notation), we know that

$$M/\mathcal{U} = \prod_{i \in I} M_i/\mathcal{U} \text{ is } L\text{-isomorphic to } \varinjlim (M_{|J}; \{\pi_{KJ} : J \subseteq K, J \in \mathcal{U}\}).$$

We shall use this fact to build a  $L$ -morphism  $\gamma^U$  such that

$$\gamma^U \circ (\nu_I \circ \iota) = Id_P,$$

then the demonstration will be finished. As  $\mathcal{U}$  will remain fixed through the proof, we will indicate  $\gamma^U$  just by  $\gamma$ . As the proof is a little bit long and technical it will be carry through with the aid of several Facts. We will make free usage of the notational conventions in 5 and 17.



For each  $J \in \mathcal{U}$ ,  $i \in I$ ,  $\bar{x} \in M_{|J} = \prod_{j \in J} M_j$  and  $y \in M_i$  we define

$$V_{J,i}(\bar{x}, y) = \{j \in J \cap i^\rightarrow : f_{ji}(x_j) = y\}.$$

**Fact 19** For each  $J \in \mathcal{U}$ ,  $i \in I$ ,  $\bar{x} \in M_{|J}$  and  $y, z \in M_i$ ,

a)  $z \neq y \Rightarrow V_{J,i}(\bar{x}, y) \cap V_{J,i}(\bar{x}, z) = \emptyset.$

b)  $J \cap i^\rightarrow = \coprod_{y \in M_i} V_{J,i}(\bar{x}, y).$  <sup>6</sup>

*Proof.* Item (a) follows immediately from the fact that  $f_{ji}$  is a function. For (b), by the definition of  $V_{J,i}(\bar{x}, y)$  it is clearly enough to show that the left side of the equality is contained in the right side, but note that if  $j \in J \cap i^\rightarrow$  then  $f_{ji}(x_j) \in M_i$ , as required.  $\square$

**Fact 20** For each  $J \in \mathcal{U}$  and  $i \in I$  there is a  $L$ -morphism

$$\gamma_{J,i} : M_{|J} = \prod_{k \in J} M_k \longrightarrow M_i$$

such that

a) If  $\bar{x} \in M_{|J}$  and  $y \in M_i$  then  $\gamma_{J,i}(\bar{x}) = y$  iff  $V_{J,i}(\bar{x}, y) \in \mathcal{U}.$

b) If  $J \subseteq K$  are members of  $\mathcal{U}$  and  $i \in I$  then the left diagram below commutes:

$$\begin{array}{ccc} M_{|K} & \xrightarrow{\pi_{KJ}} & M_{|J} \\ \gamma_{J,i} \searrow & & \swarrow \gamma_{K,i} \\ & M_i & \end{array} \qquad \begin{array}{ccc} M_{|J} & \xrightarrow{\gamma_{J,k}} & M_k \\ \gamma_{J,i} \searrow & & \swarrow f_{ki} \\ & M_i & \end{array}$$

c) For each  $J \in \mathcal{U}$  and  $i \leq k$  in  $I$  the right diagram below commutes.

d) For each  $k \in I$ ,  $\gamma_{I,k} \circ \iota = \pi_k \circ \iota$ , where  $\pi_k : M \longrightarrow M_k$  is the canonical projection.

*Proof.* Because  $\mathcal{U}$  is a directed filter in  $\langle I, \leq \rangle$  (item 2.2) for each  $J \in \mathcal{U}$  and  $i \in I$  we have  $J \cap i^\rightarrow \in \mathcal{U}$ ; because  $\mathcal{U}$  is an ultrafilter and  $M_i$  is finite, the Fact 19.(b) implies that there is a unique  $y \in M_i$  such that  $V_{J,i}(\bar{x}, y) \in \mathcal{U}$ . We define

$$\gamma_{J,i}(\bar{x}) = \text{the unique } y \in M_i \text{ such that } V_{J,i}(\bar{x}, y) \in \mathcal{U}.$$

It is clear that the item (a) is verified. Now, we must show that  $\gamma_{J,i}$  is a  $L$ -morphism. To make easier the reading, if  $J \in \mathcal{U}$ , we will indicate the symbols interpretations of  $L$  in  $M_{|J}$  by an exponent  $J$ . Then, if  $c$  is a constant symbol in  $L$ , we will use  $c^J$  instead  $c^{M_{|J}}$ ; analogously for the functional and relational symbols.

\* Let  $c \in ct(L)$ . We saw in 5 (item 3.1) that  $c^J$  is a sequence  $\langle c^{M_j} \rangle \in M_{|J}$ . So, as the  $f_{ji}$  are  $L$ -morphisms, we get

$$V_{J,i}(c^J, c^{M_i}) = \{j \in J \cap i^\rightarrow : f_{ji}(c_j^J) = c^{M_i}\} = \{j \in J \cap i^\rightarrow : f_{ji}(c^{M_j}) = c^{M_i}\} = J \cap i^\rightarrow$$

that belongs to  $\mathcal{U}$ . By item (a),  $\gamma_{J,i}(c^J) = c^{M_i}$ , as we wish.

\* Let  $\omega$  be a  $n$ -ary functional symbol in  $L$ . If  $\bar{x}_1, \dots, \bar{x}_n \in (M_{|J})^n$  and  $j \in J$  then, by 5, we have

(A)  $\omega^J(\bar{x}_1, \dots, \bar{x}_n)(j) = \omega^{M_j}(x_{1j}, \dots, x_{nj}).$

Consider

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<sup>6</sup> $\coprod$  indicates that this union is *disjunctive*.

$$\begin{cases} y^p &= \gamma_{J,i}(\bar{x}_p), & 1 \leq p \leq n; \\ z &= \omega^{M_i}(y^1, \dots, y^n); \\ h &= \omega^J(\bar{x}_1, \dots, \bar{x}_n) & (\in M_{|J}). \end{cases}$$

We will show that

$$(B) \quad \bigcap_{p=1}^n V_{J,i}(\bar{x}_p, y^p) \subseteq V_{J,i}(h, z).$$

If  $j \in \bigcap_{p=1}^n V_{J,i}(\bar{x}_p, y^p)$  then the definition of  $V_{J,i}$  implies

$$(C) \quad \forall 1 \leq p \leq n, \quad f_{ji}(x_{pj}) = y^p.$$

Because the  $f_{ji}$  are  $L$ -morphisms, (A) and (C) give

$$f_{ji}(h_j) = f_{ji}(\omega^{M_j}(x_{1j}, \dots, x_{nj})) = \omega^{M_i}(f_{ji}(x_{1j}), \dots, f_{ji}(x_{nj})) = \omega^{M_i}(y^1, \dots, y^n) = z,$$

and this proves (B). As the intersection of the left side in (B) belongs to  $\mathcal{U}$  we have  $V_{J,i}(h, z) \in \mathcal{U}$ . By the item (a) of this Fact, this means that

$$\gamma_{J,i}(\omega^J(\bar{x}_1, \dots, \bar{x}_n)) = \omega^{M_i}(\gamma_{J,i}(\bar{x}_1, \dots, \bar{x}_n))$$

showing that  $\gamma_{J,i}$  preserves the operation  $\omega$ ;

\* Let  $R$  be a  $n$ -ary relational symbol in  $L$ . Consider  $\bar{x}_1, \dots, \bar{x}_n \in (M_{|J})^n$ . By 5

$$(D) \quad M_{|J} \models R[\bar{x}_1, \dots, \bar{x}_n] \text{ iff } \forall j \in J, \quad M_j \models R[x_{1j}, \dots, x_{nj}].$$

As above, let  $y^p = \gamma_{J,i}(\bar{x}_p)$ ,  $1 \leq p \leq n$ . We must show that

$$(E) \quad M_{|J} \models R[\bar{x}_1, \dots, \bar{x}_n] \Rightarrow M_i \models R[y^1, \dots, y^n].$$

Because  $\bigcap_{p=1}^n V_{J,i}(\bar{x}_p, y^p) \in \mathcal{U}$ , this intersection is non-empty; if  $j$  is a member of this intersection, the topic (C) above is checked. Then, it follows from (D) and the fact that  $f_{ji}$  is a  $L$ -morphism that

$$M_{|J} \models R[\bar{x}_1, \dots, \bar{x}_n] \Rightarrow M_j \models R[x_{1j}, \dots, x_{nj}] \Rightarrow M_i \models R[f_{ji}(x_{1j}), \dots, f_{ji}(x_{nj})],$$

with this and (C) we obtain (E), completing the proof that  $\gamma_{J,i}$  is a  $L$ -morphism.

b) Let  $\bar{t} \in M_{|K}$  and  $\bar{x} = \pi_{KJ}(\bar{t})$ <sup>7</sup>. If  $y = \gamma_{J,i}(\bar{x})$  we will see that

$$V_{J,i}(\bar{x}, y) \subseteq V_{K,i}(\bar{t}, y).$$

In fact, if  $j \in V_{J,i}(\bar{x}, y)$  (obviously contained  $K \cap i^\rightarrow$ ) then

$$f_{ji}(t_j) = f_{ji}(x_j) = y,$$

as required. As  $V_{J,i}(\bar{x}, y) \in \mathcal{U}$  we have  $V_{K,i}(\bar{t}, y) \in \mathcal{U}$  and the item (a) ensures that  $\gamma_{K,i}(\bar{t}) = y = \gamma_{J,i}(\pi_{KJ}(\bar{t}))$ , as we need.

c) Let  $\bar{x} \in M_{|J}$  and  $z = \gamma_{J,k}(\bar{x})$ . Then

$$(F) \quad V_{J,k}(\bar{x}, z) \subseteq V_{J,i}(\bar{x}, f_{ki}(z)).$$

In fact, if  $j \in V_{J,k}(\bar{x}, z)$  (contained in  $J \cap i^\rightarrow$  because  $i \leq k$ ) then  $f_{jk}(x_j) = z$ . As  $\mathcal{M}$  is a cofiltered diagram, we have

$$f_{ji}(x_j) = f_{ki}(f_{jk}(x_j)) = f_{ki}(z)$$

showing that  $j \in V_{J,i}(\bar{x}, f_{ki}(z))$ ; as the topic (F) above ensures that this set belongs to  $\mathcal{U}$ , the item (a) implies  $\gamma_{J,i} = f_{ki} \circ \gamma_{J,k}$ , as needed.

d) For each  $\bar{x} \in P$  and  $k \in I$  observe that  $\pi_k(\iota(\bar{x})) = x_k$ . It follows from the relation (b) in 17 (page 8) that

$$V_{I,k}(\iota(\bar{x}), x_k) = \{j \in k^\rightarrow : f_{jk}(x_j) = x_k\} = k^\rightarrow.$$

Because  $\mathcal{U}$  is a directed ultrafilter, we have  $V_{I,k}(\iota(\bar{x}), x_k) \in \mathcal{U}$  and the item (a) gives the needed conclusion, closing the proof of the Fact 20.  $\square$

By Proposition 14 we have

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<sup>7</sup>We recall that  $\pi_{KJ}$  is the projection that forgets the coordinates out of  $K$ .

$$M/\mathcal{U} = \varinjlim (M|_J, \{\pi_{KJ} : J \subseteq K, J \in \mathcal{U}\}) .$$

Fact 20.(b) and the universal property of the filtered colimits ensures that, for each  $i \in I$ , there is a unique  $L$ -morphism,  $\gamma_i : M/\mathcal{U} \longrightarrow M_i$ , such that for all  $J \in \mathcal{U}$  the left diagram below commutes:

$$(*) \quad \begin{array}{ccc} M|_J & \xrightarrow{\nu_J} & M/\mathcal{U} \\ & \searrow \gamma_{J,i} & \swarrow \gamma_i \\ & M_i & \end{array} \quad \begin{array}{ccc} M/\mathcal{U} & \xrightarrow{\gamma_k} & M_k \\ & \searrow \gamma_i & \swarrow f_{ki} \\ & M_i & \end{array}$$

**Fact 21** For each  $i \leq k$  in  $I$ , the right diagram above in  $(*)$  is commutative.

*Proof.* For each  $i \leq k$  in  $I$  and  $J \in \mathcal{U}$ , the Fact 20.(c) gives  $\gamma_{J,i} = f_{ki} \circ \gamma_{J,k}$ . Then, the commutativity of the left diagram above in  $(*)$  – for  $k$  and  $i$  –, implies that, for all  $J \in \mathcal{U}$  we have

$$f_{ki} \circ \gamma_k \circ \nu_J = f_{ki} \circ \gamma_{J,k} = \gamma_{J,i} = \gamma_i \circ \nu_J.$$

Now, the uniqueness of the  $\gamma_i$  that make the left diagram commutative, for all  $J \in \mathcal{U}$  ensures that  $f_{ki} \circ \gamma_k = \gamma_i$ , as required.  $\square$

Fact 21 shows that  $(M/\mathcal{U}, \{\gamma_i : i \in I\})$  is a cone over the cofiltered diagram  $\mathcal{M}$ . Then the universal property of the cofiltered limits ensures that there is a unique  $L$ -morphism

$$(**) \quad \gamma : M/\mathcal{U} \longrightarrow P = \varprojlim \mathcal{M} \quad \begin{array}{ccc} M/\mathcal{U} & \xrightarrow{\gamma} & P \\ & \searrow \gamma_i & \swarrow \lambda_i \\ & M_i & \end{array}$$

such that for all  $i \in I$  the diagram above commutes.

We will check now that

$$(G) \quad \gamma \circ \nu_I \circ \iota = Id_P.$$

As  $(P, \{\lambda_i : i \in I\}) = \varprojlim \mathcal{M}$ , the universal property of the limits ensures that to prove (G) it is enough to show that for all  $k \in I$

$$(H) \quad \lambda_k \circ (\gamma \circ \nu_I \circ \iota) = \lambda_k.$$

$$\text{As } \left\{ \begin{array}{ll} \lambda_k \circ \gamma &= \gamma_k \quad \text{by the diagram in } (**); \\ \gamma_k \circ \nu_I &= \gamma_{I,k} \quad \text{by the left diagram in } (*), \text{ page 11}; \\ \pi_k \circ \iota &= \lambda_k \quad \text{by } (\sharp) \text{ in 17, page 8,} \end{array} \right.$$

(H) is equivalent to  $\gamma_{I,k} \circ \iota = \pi_k \circ \iota$ , but that is precisely the content of the Fact 20.(d), so the proof of the Theorem is complete.  $\square$

$$\begin{array}{ccccc}
P & \xrightarrow{\iota} & M & \xrightarrow{\nu_I} & M/\mathcal{U} \\
\downarrow \lambda_k & & \nearrow \gamma_{I,k} & & \downarrow \gamma \\
& & & \nearrow \gamma_k & \\
& & & & \downarrow \gamma \\
M_k & \xleftarrow{\lambda_k} & P & & 
\end{array}$$

The Lemma 4 and the Theorem 18 produce the following

**Corollary 22** *Let  $\mathcal{A} = \text{Mod}(T)$  where  $T$  is a theory axiomatized by  $L$ -sentences of the form  $\forall \vec{x}(\psi_0(\vec{x}) \rightarrow \psi_1(\vec{x}))$  where  $\psi_0(\vec{x}), \psi_1(\vec{x})$  are formulas in  $\exists^+(L)$ . Then the full subcategory  $\mathcal{A} \subseteq \mathbf{L}\text{-mod}$  has profinite objects<sup>8</sup>, that is,  $\mathcal{A}$  is closed in  $\mathbf{L}\text{-mod}$  under the formation of such limits.*  $\square$

## 5 Final Remarks

As an application of the results above we mention the case of the *Special Groups*, a first-order axiomatization of the algebraic theory of quadratic forms (see [DM2]). The suitable first-order language,  $L_{SG}$ , contains two symbols for constants (1 and -1), one symbol for binary operation (multiplication) and one symbol for quaternary relation ( $\equiv$ , the isometry between quadratic forms with dimension 2). The special groups axioms (Definition 1.2 in [DM2]) have the form  $\forall \vec{x}(\psi_0(\vec{x}) \rightarrow \psi_1(\vec{x}))$ , where  $\psi_0(\vec{x}), \psi_1(\vec{x})$  are existential-positives  $L_{SG}$ -formulas, from the results 18 and 22 above we can conclude that there are profinite special groups and that they are retracts of ultraproducts of finite special groups, a result contained in Theorem 5.8 in [Mrn1] and that has further consequences (e.g., in the forthcoming [MDM]).

As we mentioned before, our main motivation in [Mrn1] was to study the class of profinite special groups and particularly the construction of the Profinite Hull of Special Groups functor (an English language version of those main results will appear in [MM]; see also the forthcoming [Mrn2] for further application of such construction). The perception that some of those constructions and results can be transported toward the general context of  $L$ -structures has appeared with the results of the present work. Further material about the "Model Theory of Profinite Structures" are being elaborated in [Mrn3].

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<sup>8</sup>We note that  $\mathcal{A}$  has already some finite object:  $\mathbf{1}$ , the final object of  $\mathbf{L}\text{-mod}$ , belongs to  $\mathcal{A}$ , as we have noted in 5.

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